Continuity equation (mass conservation)

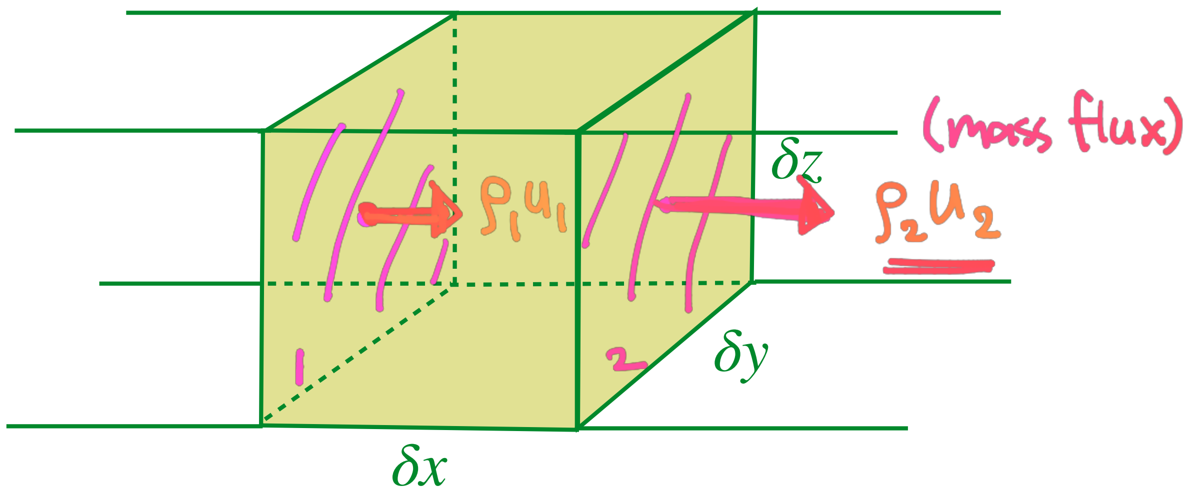
1. Eulerian derivation
2. Lagrangian derivation

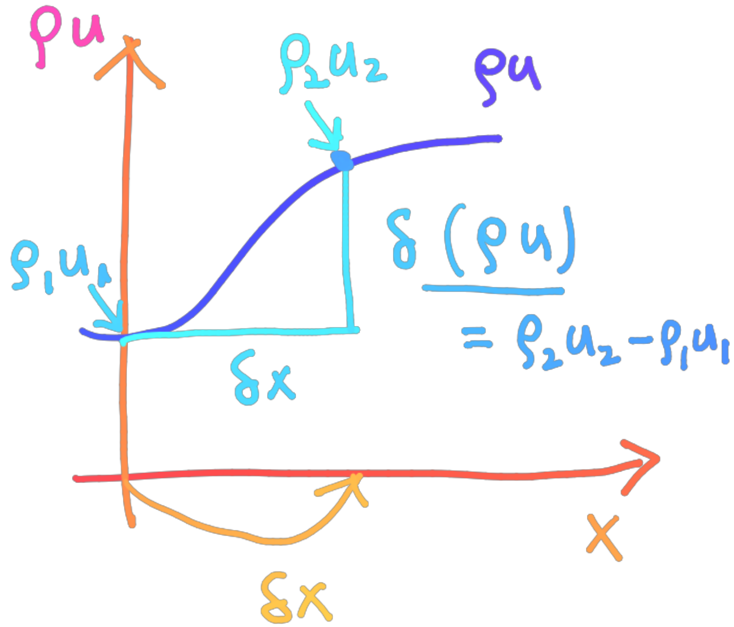
$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0$$

$$\frac{D\rho}{Dt} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0$$

Image from flickr (through pinterist)

Eulerian concept





$m = \rho\,\delta x\,\delta y\,\delta z$

$V = \delta x\,\delta y\,\delta z$

$\boxed{\text{In x-dir}}$

$$\frac{\Delta m}{\Delta t} = -(\rho_2 u_2 - \rho_1 u_1)\,\delta y\,\delta z$$

$$= \delta x\,\delta y\,\delta z \cdot \frac{\Delta \rho}{\Delta t}$$

(because $\delta x, \delta y, \delta z$ are fixed Eulerian!!)

$$\Rightarrow \lim_{\Delta t, \delta x \to 0} \frac{\Delta \rho}{\Delta t} = -\frac{\delta(\rho u)}{\delta x}$$

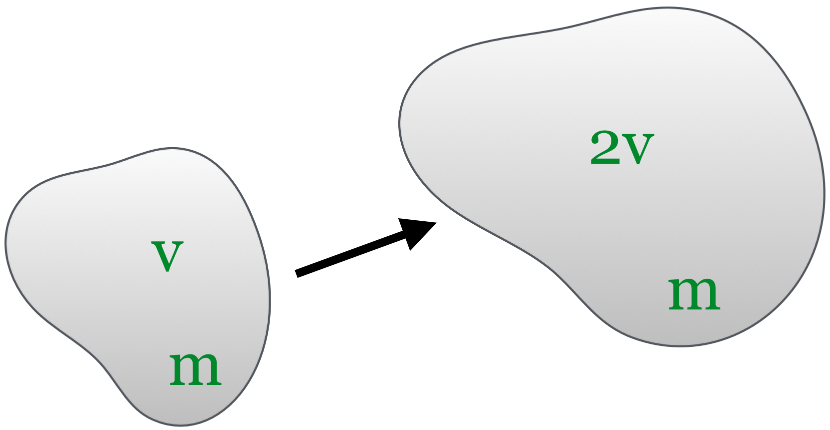
$$\Downarrow$$

$$\frac{\partial \rho}{\partial t} = -\frac{\partial(\rho u)}{\partial x} \underbrace{- \frac{\partial(\rho v)}{\partial y} - \frac{\partial(\rho w)}{\partial z}}_{\text{the same way}}$$

$$\frac{\partial \rho}{\partial t} + \underbrace{\frac{\partial(\rho u)}{\partial x}}_{x} + \underbrace{\frac{\partial(\rho v)}{\partial y}}_{y} + \underbrace{\frac{\partial(\rho w)}{\partial z}}_{z} = 0$$

divergence in x y z direction

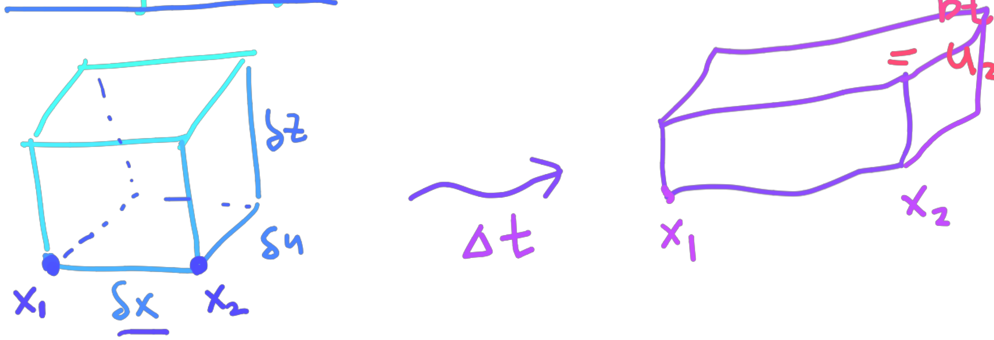
Lagrangian concept



$$\underline{m = \rho\,\delta x\,\delta y\,\delta z}$$

$$\delta x = x_2 - x_1$$

$$\frac{D\delta x}{Dt} = \frac{D(x_2 - x_1)}{Dt} = \frac{Dx_2}{Dt} - \frac{Dx_1}{Dt} = u_2 - u_1 = \delta u$$



$$\frac{\frac{Dm}{Dt}}{m} = \frac{\frac{D(\rho\,\delta x\,\delta y\,\delta z)}{Dt}}{\rho\,\delta x\,\delta y\,\delta z} = \frac{\frac{D\rho}{Dt}\cdot\delta x\,\delta y\,\delta z + \rho\frac{D\delta x}{Dt}\cdot\delta y\,\delta z + \rho\,\delta x\frac{D\delta y}{Dt}\cdot\delta z + \rho\,\delta x\,\delta y\frac{D\delta z}{Dt}}{\rho\,\delta x\,\delta y\,\delta z}$$

$$// \ \emptyset$$

$$\emptyset = \frac{1}{\rho}\frac{D\rho}{Dt} + \frac{1}{\delta x}\frac{D\delta x}{Dt} + \frac{1}{\delta y}\frac{D\delta y}{Dt} + \frac{1}{\delta z}\frac{D\delta z}{Dt}$$

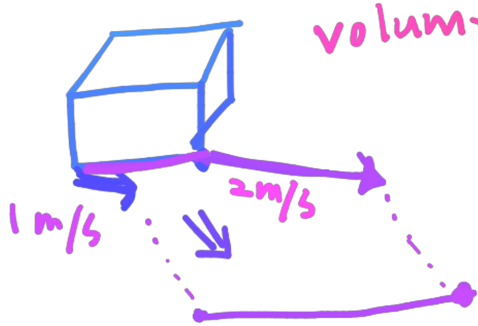
$$\lim_{\delta\to\emptyset} \quad \emptyset = \frac{1}{\rho}\frac{D\rho}{Dt} + \frac{\delta u}{\delta x} + \frac{\delta v}{\delta y} + \frac{\delta w}{\delta z}$$

$$\boxed{\frac{D\rho}{Dt} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0}$$

$$\frac{D\rho}{Dt} + \rho\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)=0$$

$$\frac{1}{\rho}\frac{D\rho}{Dt} = -\underline{\nabla\cdot\vec{V}}$$

divergence of volume !!



$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z} = 0$$

$$+\rho\frac{\partial u}{\partial x} + \rho\frac{\partial v}{\partial y} + \rho\frac{\partial w}{\partial z}$$

$$= \frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} + \frac{\partial (\rho w)}{\partial z} = 0$$

$$\Downarrow$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} + \frac{\partial (\rho w)}{\partial z} = 0$$

$$\underline{\frac{\partial \rho}{\partial t}} = -\underline{\nabla\cdot\left(\rho\vec{V}\right)}$$

divergence of mass flux !!

They are same!