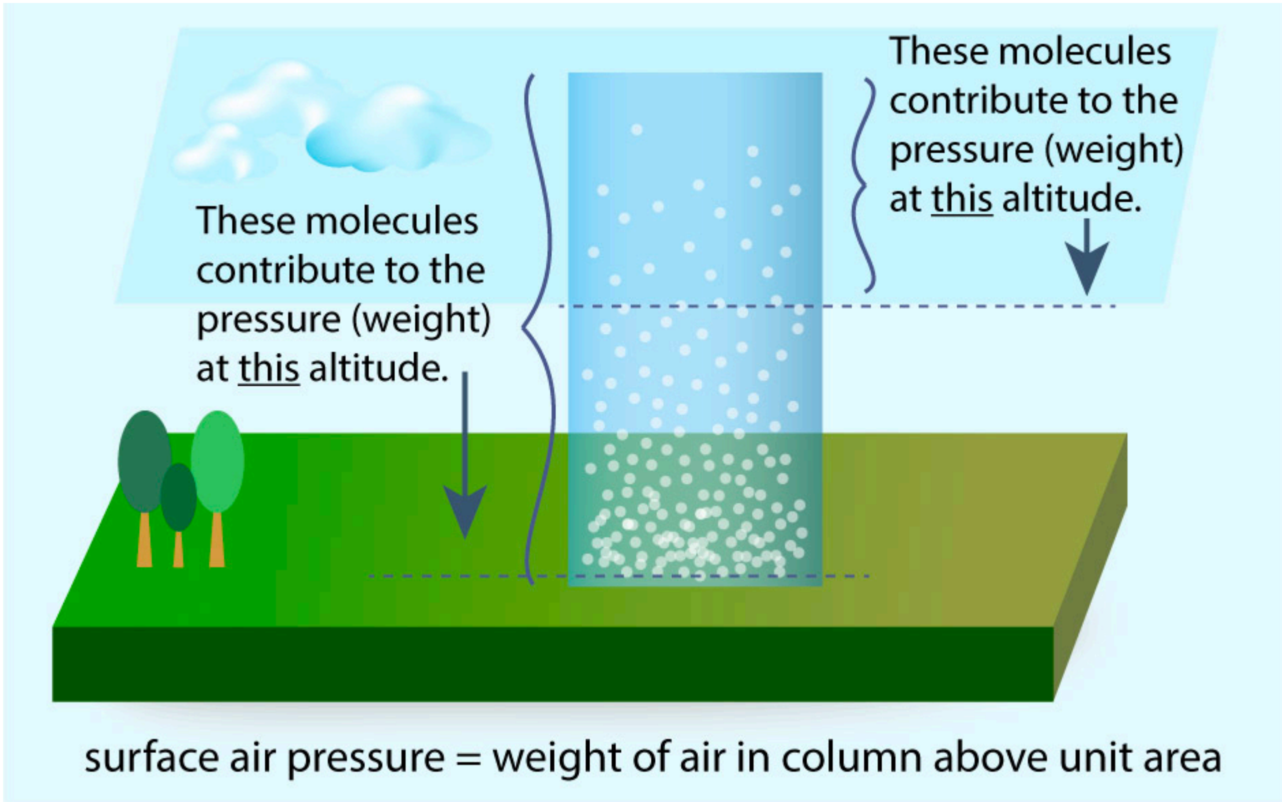
Pressure coordinates

1. Review: Primitive equations in height (z) coordinates
2. Pressure coordinates
3. Primitive equations in pressure (p) coordinate
4. Log-pressure coordinate



From liberaldictionary.com

Primitive equations in z-coords

Six equations with six known (u, v, w, T, p, ρ)

Mathematically closed; and **solvable**

$$\frac{Du}{Dt} - fv = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu \nabla^2 u$$

$$\frac{Dv}{Dt} + fu = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu \nabla^2 v$$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g$$

Momentum (velocity)

where

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$$

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$\nabla \cdot \vec{u} \equiv \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \vec{u} = 0$$

Continuity (mass)

$$c_p \frac{DT}{dt} - \alpha \frac{Dp}{dt} = J$$

Thermodynamic energy (temperature)

$$p = \rho R T$$

State

z-coords ➜ p-coords (1)

Pressure can be used as a vertical coordinate
due to strong hydrostatic balance: z ➔ p, w ➔ ω

> Cartesian (z-coordinates)

$$u = u(x, y, z, t)$$

$$\frac{Du}{Dt} \equiv \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$$

> P-coordinates

$$u = u(x, y, p, t)$$

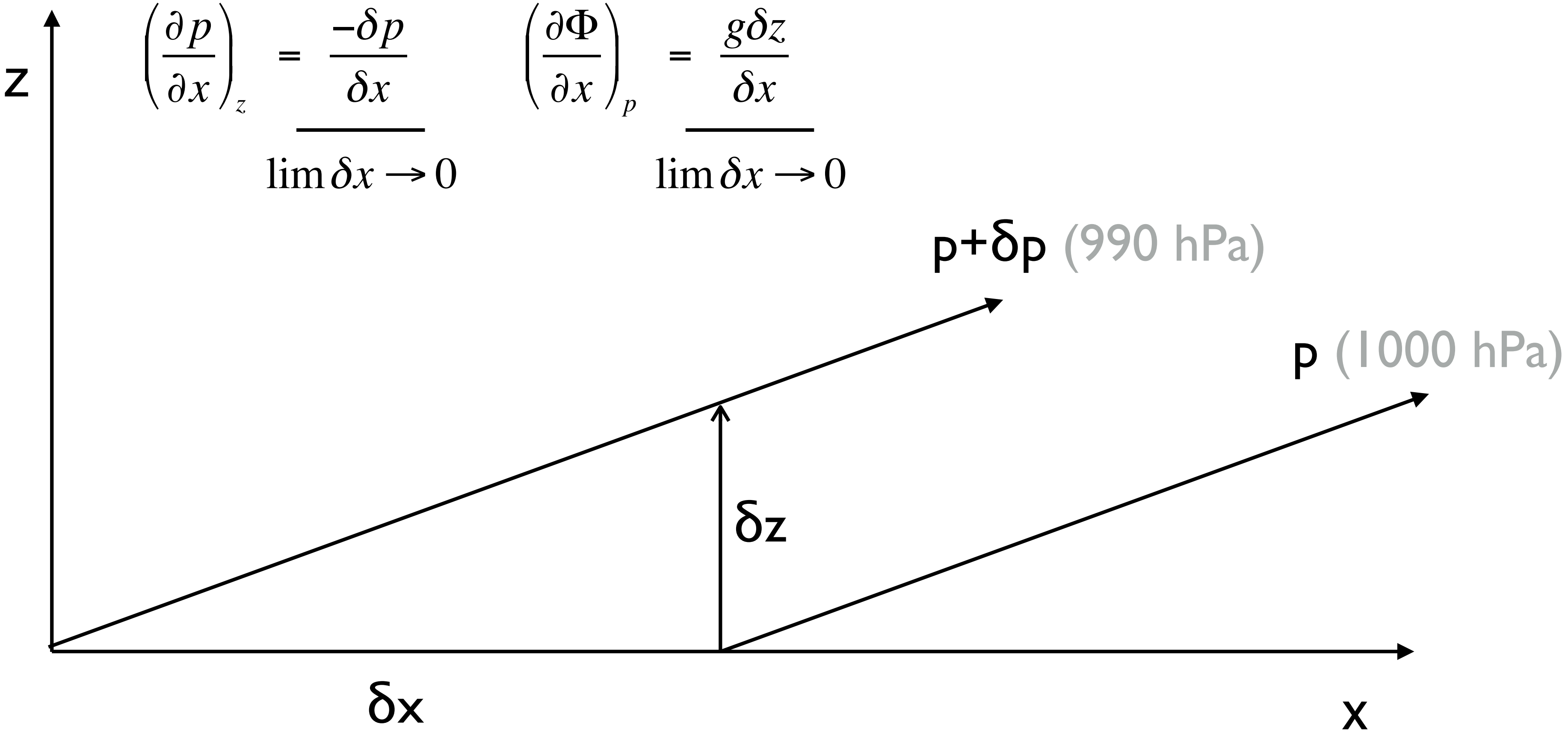
$$\frac{Du}{Dt} \equiv \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \omega\frac{\partial u}{\partial p}$$

z-coords ➜ p-coords (2)

Pressure can be used as a vertical coordinate due to strong hydrostatic balance: $\frac{1}{\rho}\frac{\partial p}{\partial x} \rightarrow \frac{\partial \Phi}{\partial x}$

$$\left(\frac{\partial p}{\partial x}\right)_z = \frac{-\delta p}{\delta x} \quad \lim \delta x \to 0$$

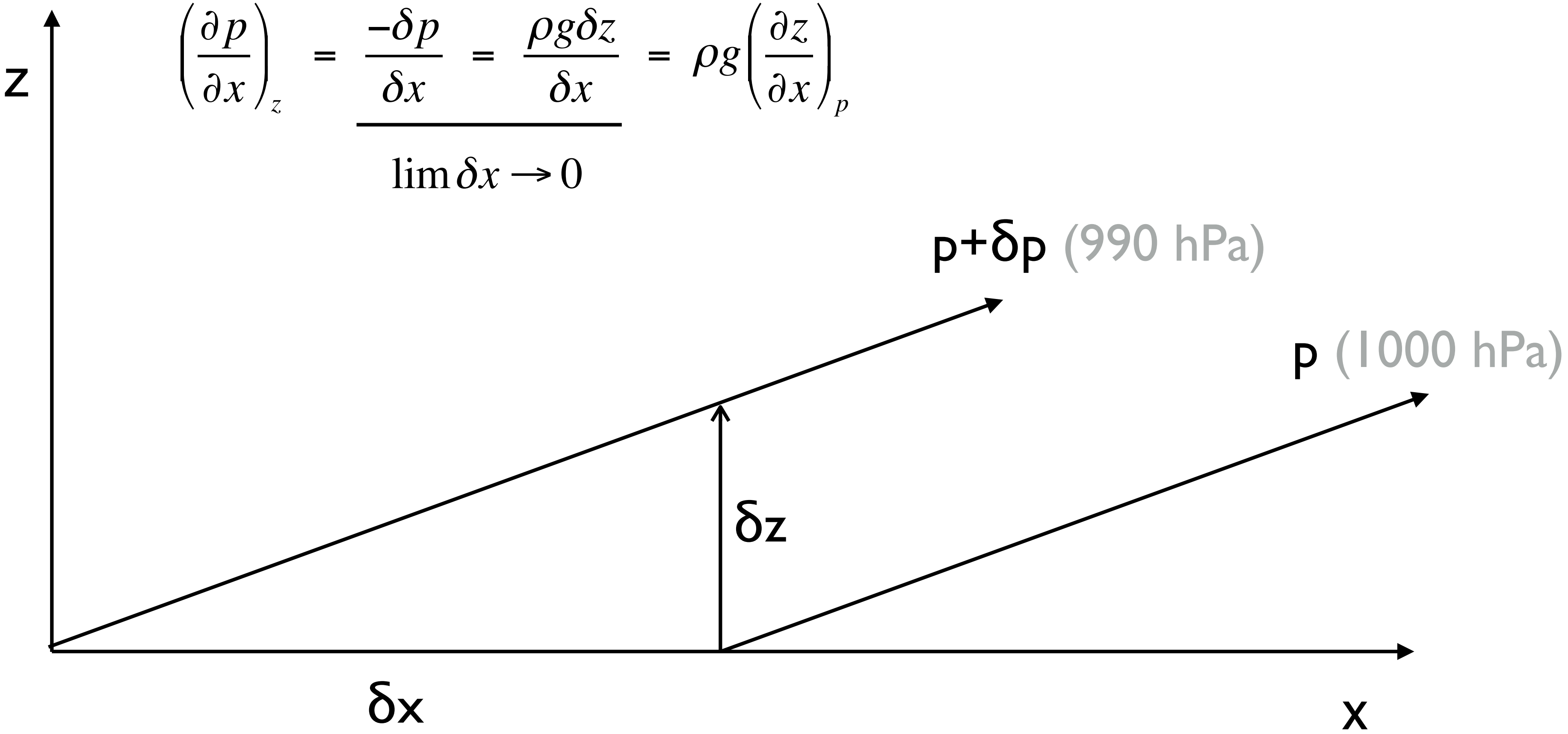
$$\left(\frac{\partial \Phi}{\partial x}\right)_p = \frac{g\delta z}{\delta x} \quad \lim \delta x \to 0$$



z-coords ➜ p-coords (2)

Pressure can be used as a vertical coordinate due to strong hydrostatic balance: $\frac{1}{\rho}\frac{\partial p}{\partial x} \rightarrow \frac{\partial \Phi}{\partial x}$

$$\left(\frac{\partial p}{\partial x}\right)_z = \underbrace{\frac{-\delta p}{\delta x} = \frac{\rho g \delta z}{\delta x}}_{\lim \delta x \to 0} = \rho g \left(\frac{\partial z}{\partial x}\right)_p$$



z-coords ➜ p-coords (3)

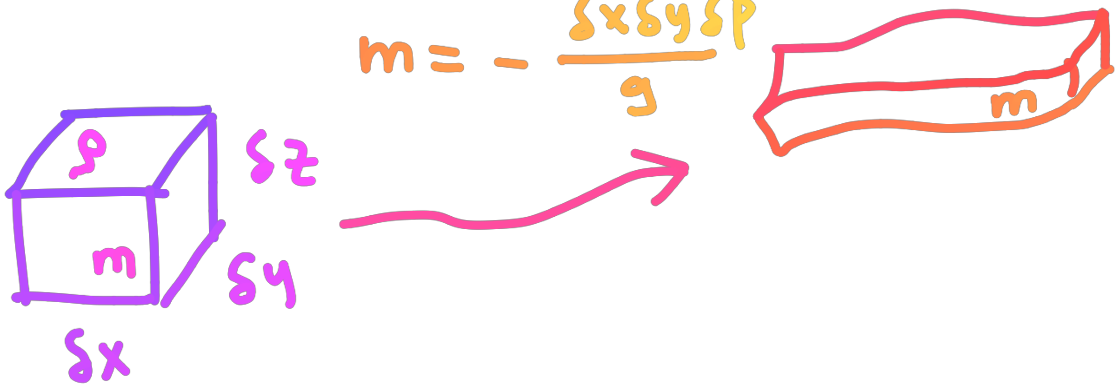
Continuity equation can be re-derived and simplified
: **Prognostic → Diagnostic!**

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \omega}{\partial p} = 0$$

$$-\frac{1}{g}\delta p = +\rho\,\delta z$$

$$m = \rho\,\delta x\,\delta y\,\delta z$$

$$m = -\frac{\delta x\,\delta y\,\delta p}{g}$$



$$0 = \frac{\frac{Dm}{Dt}}{m} = \frac{-\frac{1}{g}\frac{D\,\delta x\,\delta y\,\delta p}{Dt}}{-\frac{1}{g}\delta x\,\delta y\,\delta p} = \frac{1}{\delta x}\frac{D\delta x}{Dt} + \frac{1}{\delta y}\frac{D\delta y}{Dt} + \frac{1}{\delta p}\frac{D\delta p}{Dt}$$

$$= \frac{\delta u}{\delta x} + \frac{\delta v}{\delta y} + \frac{\delta \omega}{\delta p}$$

z-coords ➜ p-coords (4)

Thermodynamics energy equation can further be organized because the physical meaning of **Dp/Dt** becomes more clear

$$c_p \frac{DT}{Dt} - \alpha \frac{Dp}{Dt} = J$$

$$\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} - S_p \omega = \frac{J}{c_p}$$

$$c_p \frac{DT}{Dt} - \alpha \omega = J$$

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} + \omega \frac{\partial T}{\partial p}$$

$$\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} - \underbrace{\left(\frac{\alpha}{c_p} - \frac{\partial T}{\partial p} \right)}_{\equiv S_p} \omega = \frac{J}{c_p}$$

z-coords ➜ p-coords (4)

Thermodynamics energy equation can further be organized because the physical meaning of **Dp/Dt** becomes more clear

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} - S_p\omega = \frac{J}{c_p}$$

$$S_p \equiv \frac{\alpha}{c_p} - \frac{\partial T}{\partial p} = -\frac{T}{\theta}\frac{\partial \theta}{\partial p} \qquad S_p = (\Gamma_d - \Gamma)/\rho g$$

$$\rho g \left(S_p = \frac{\alpha}{c_p} - \frac{\partial T}{\partial p} \right) \qquad \left(\alpha = \frac{1}{\rho} \right)$$

$$\rho g S_p = \frac{g}{c_p} - \rho g \frac{\partial T}{\partial p} \leftarrow \frac{\partial T}{\partial z} = -\Gamma$$

$$\rho g S_p = \Gamma_d - \Gamma$$

Primitive equations in p-coords

Five equations with five known (u, v, ω, T, Φ)

Mathematically closed; and **solvable**

$$\frac{Du}{Dt} - fv = -\frac{\partial \Phi}{\partial x} + F_x$$

$$\frac{Dv}{Dt} + fu = -\frac{\partial \Phi}{\partial y} + F_y$$

Momentum (velocity)

$$\frac{\partial \Phi}{\partial p} = -\frac{RT}{p}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \omega}{\partial p} = 0$$

Continuity (mass)

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} - S_p \omega = \frac{J}{c_p}$$

Thermodynamic energy (temperature)

where

$$\omega \equiv \frac{Dp}{Dt}$$

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + \omega\frac{\partial}{\partial p}$$

$$S_p \equiv \frac{1}{c_p}\frac{RT}{p} - \frac{\partial T}{\partial p} = -\frac{T}{\theta}\frac{\partial \theta}{\partial p}$$

Log-pressure vs. Height

- From hydrostatic equation (as always)

$$\frac{\partial p}{\partial z} = -\rho g \qquad \frac{\partial z}{\partial p} = -\frac{1}{g}\frac{RT}{p}$$

(Pressure-coord Version)

- Rearrange

$$\frac{\partial z}{\partial \ln p} = -\frac{RT}{g}$$

$$\delta z = -\frac{RT}{g}\delta \ln p$$

- Assuming constant atmospheric temperature

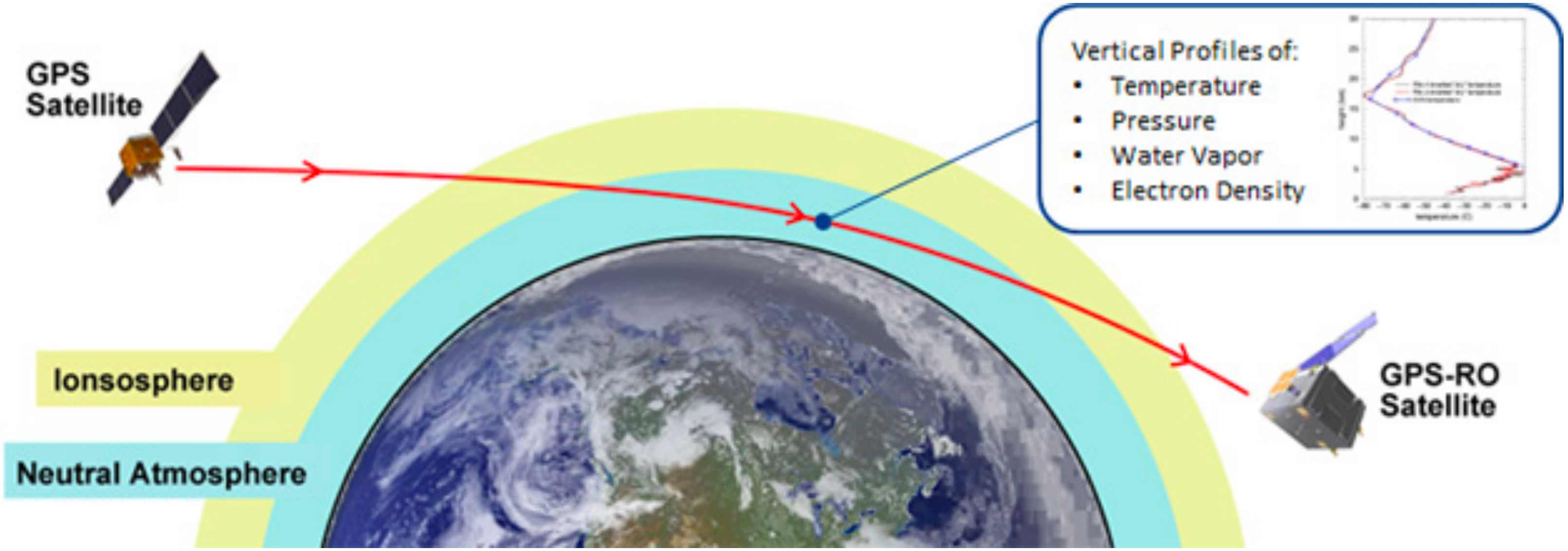
Integrate from surface to z

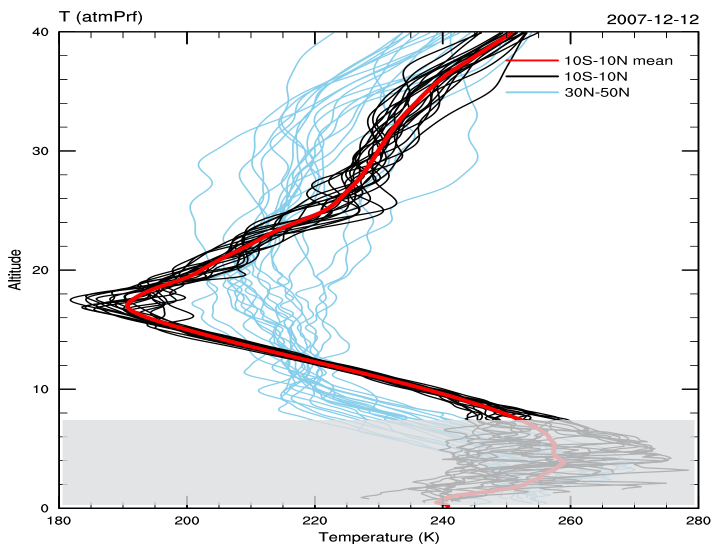
$$z^* = -\frac{R\langle T \rangle}{g}\ln(p / p_0)$$

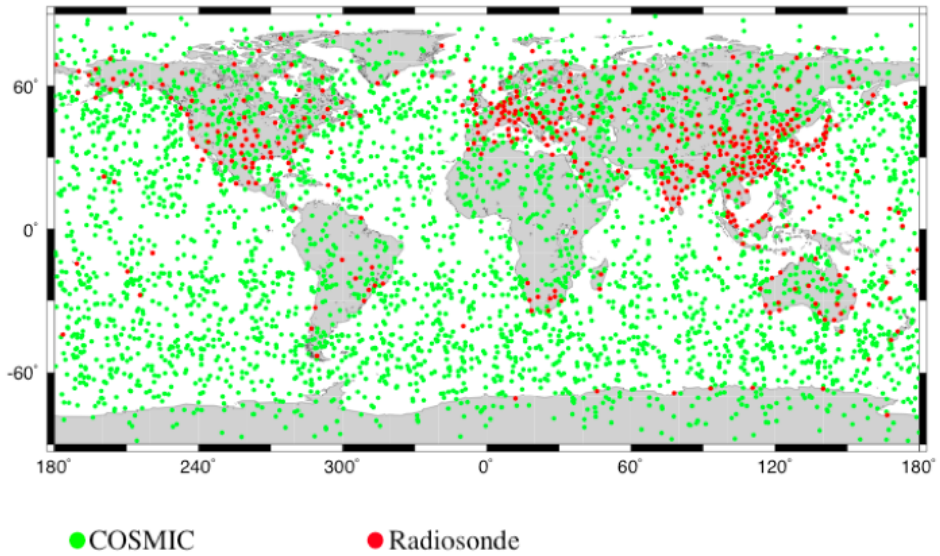
R = 287 J/Kg/K
T ~ 245 K

Log-pressure vs. Height

- GPS radio occupation







Log-pressure vs. Height