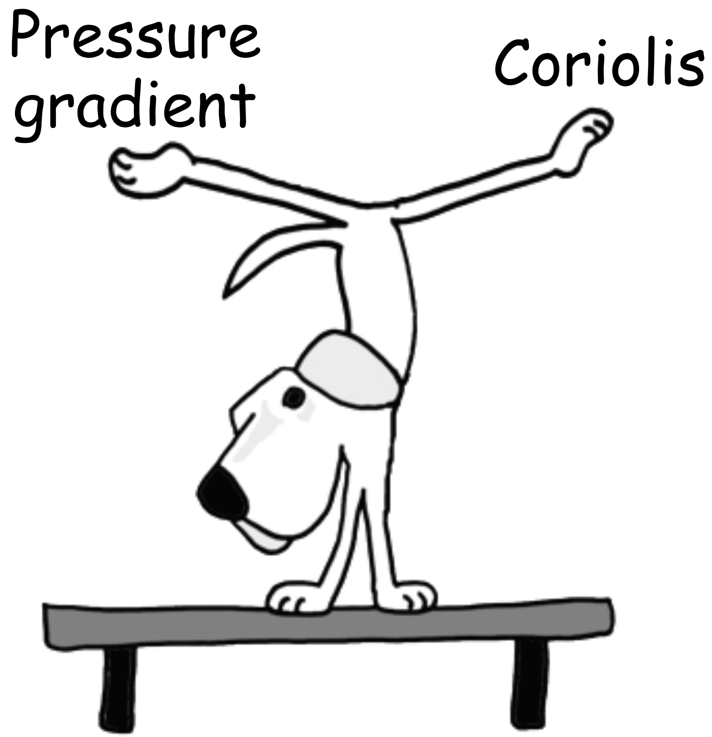
Balances in the atmosphere

Background: Equation of motion, Scale analysis

1. Geostrophic balance
2. Hydrostatic balance
3. Thermal wind balance



0. Equation of motion

$$\frac{\partial u}{\partial t} + \vec{u} \cdot \nabla u - fv = -\frac{1}{\rho}\frac{\partial p}{\partial x} + F_x$$

$$\frac{\partial v}{\partial t} + \vec{u} \cdot \nabla v + fu = -\frac{1}{\rho}\frac{\partial p}{\partial y} + F_y$$

horizontal momentum equation

$$\frac{\partial w}{\partial t} + \vec{u} \cdot \nabla w + g = -\frac{1}{\rho}\frac{\partial p}{\partial z} + F_z$$

vertical momentum equation

- Note: f (≡2Ωsinφ with angular velocity of the earth Ω)
  f* (≡2Ωcosφ) terms are neglected (traditional approximation)

0. Equation of motion (scale analysis)

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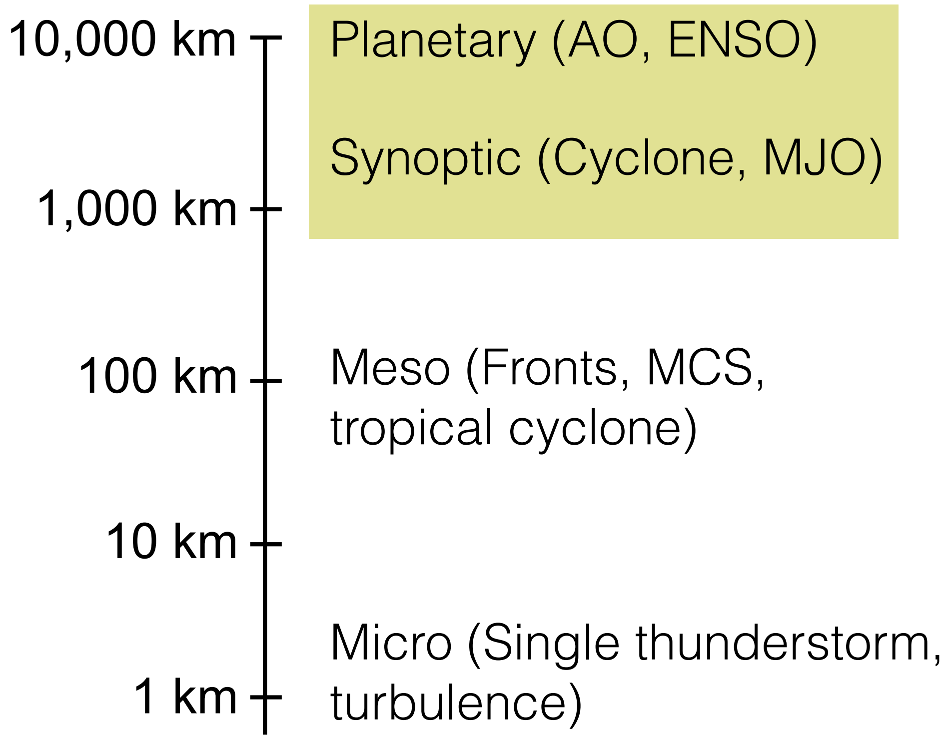
$$\frac{\partial w}{\partial t} + \vec{u}\cdot\nabla w + g = -\frac{1}{\rho}\frac{\partial p}{\partial z} + F_z$$



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U/T U²/L fU δP/Lρ

$$\frac{\partial w}{\partial t} + \vec{u} \cdot \nabla w + g = -\frac{1}{\rho}\frac{\partial p}{\partial z} + F_z$$

Typical scale (in MKS unit) of synoptic phenomena in mid-latitude

- U ~ 10 m/s
- W ~ 10^{-2} m/s
- T ~ 10^{5} s (~1 day)
- L ~ 10^{6} m
- f ~ 10^{-4} /s
- δP ~ 10^{3} Pa (N/m²; kg/ms²)
- ρ ~ 1 kg/m³

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$\sim 10^{-4}$	$\sim 10^{-4}$	$\sim 10^{-3}$	$\sim 10^{-3}$	(m/s²)

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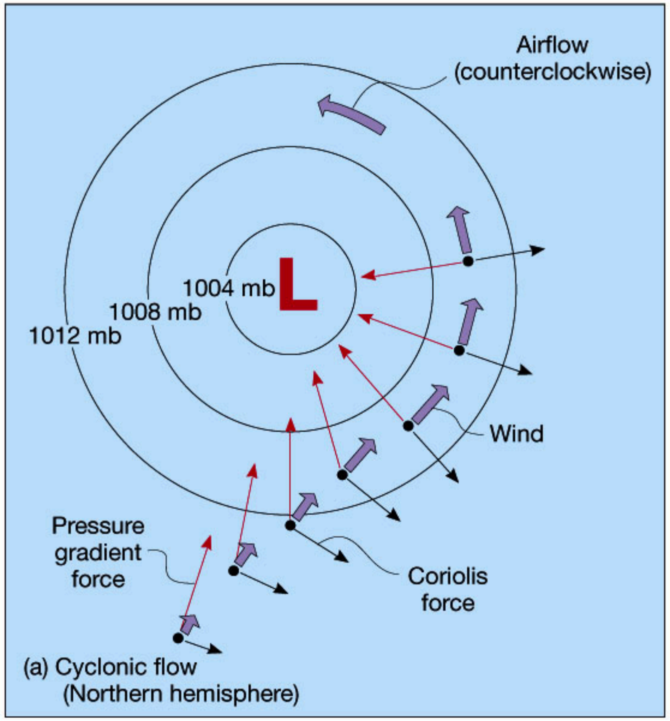
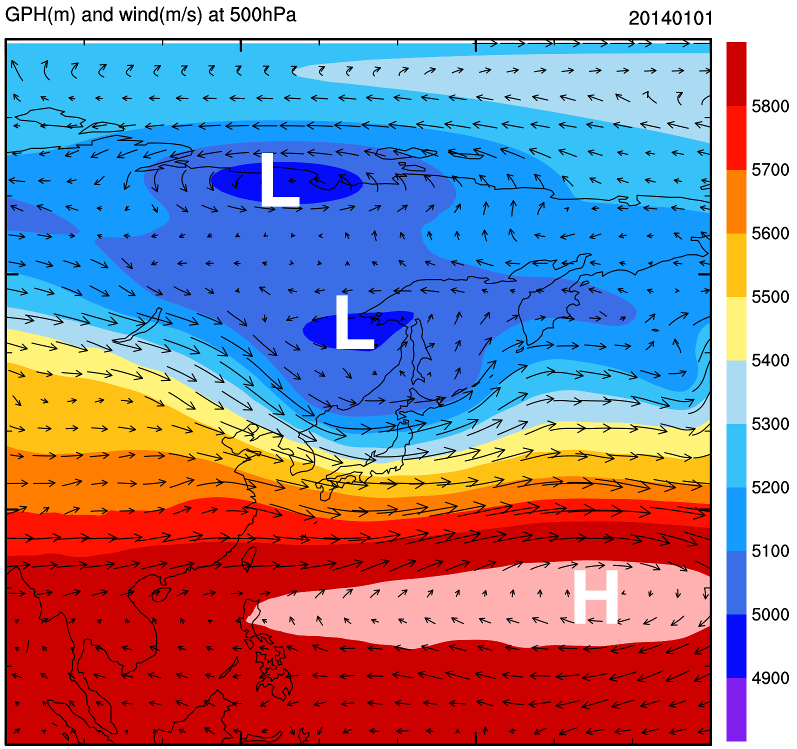
- Coriolis force and Pressure gradient force (PGF) make the first order balance

1. Geostrophic balance

$$-fv_g = -\frac{1}{\rho}\frac{\partial p}{\partial x}; \quad fu_g = -\frac{1}{\rho}\frac{\partial p}{\partial y} \quad \left(\vec{u}_g = \frac{1}{\rho f}\mathbf{k}\times\nabla_h p\right)$$



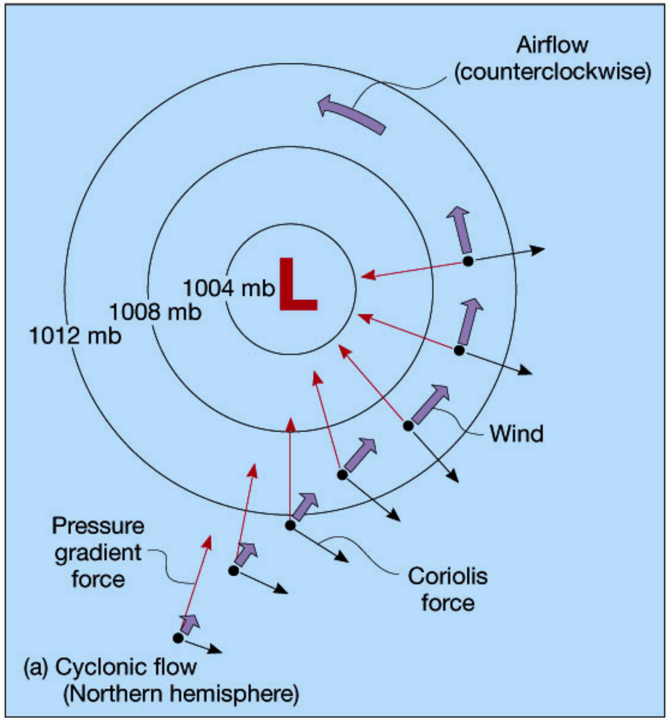
http://www.ux1.eiu.edu/~cfjps/1400/pressure_wind.html



- Coriolis force acts to the right of the wind vector in the NH
- It balances pressure gradient force ($-\nabla_h p$)

1. Geostrophic balance (p-coords)

$$-fv_g = -\frac{\partial \Phi}{\partial x}, \quad fu_g = -\frac{\partial \Phi}{\partial y}, \qquad \left(\vec{u}_g = \frac{1}{f}\mathbf{k} \times \nabla_h \Phi \right)$$



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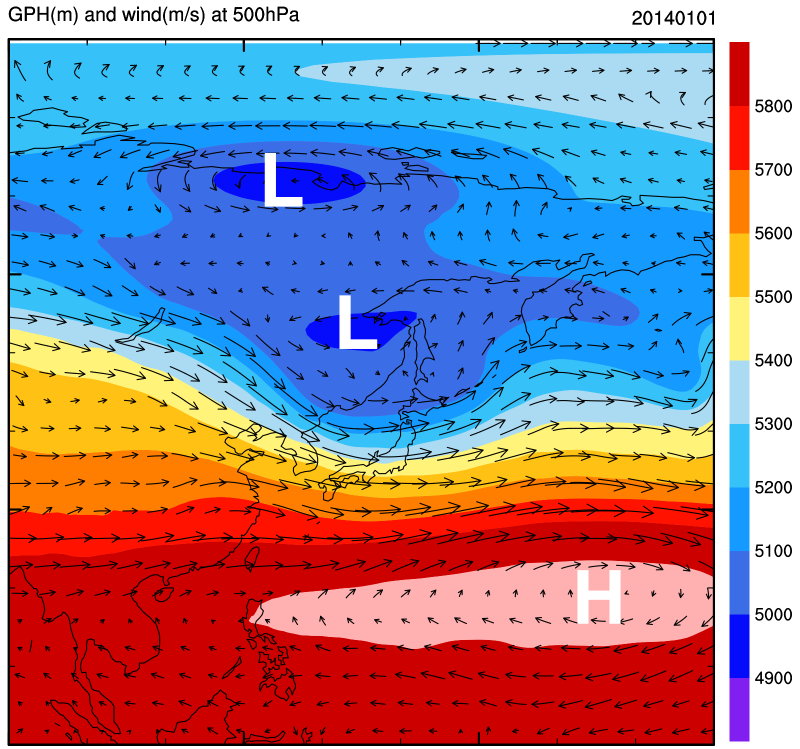

- Coriolis force acts to the right of the wind vector in the NH
- It balances pressure gradient force ($-\nabla_h p$)

2. Hydrostatic balance (scale analysis)

$$\frac{\partial u}{\partial t} + \vec{u}\cdot\nabla u - fv = -\frac{1}{\rho}\frac{\partial p}{\partial x} + F_x$$

$$\frac{\partial v}{\partial t} + \vec{u}\cdot\nabla v + fu = -\frac{1}{\rho}\frac{\partial p}{\partial y} + F_y$$

U/T	U^2/L	fU	$\delta P/L\rho$	
$\sim 10^{-4}$	$\sim 10^{-4}$	$\sim 10^{-3}$	$\sim 10^{-3}$	(m/s^2)

$$\frac{\partial w}{\partial t} + \vec{u}\cdot\nabla w + g = -\frac{1}{\rho}\frac{\partial p}{\partial z} + F_z$$

W/T	UW/L	g	$P/H\rho$

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U/T	U^2/L	fU	δP/Lρ	
~10^{-4}	~10^{-4}	~10^{-3}	~10^{-3}	(m/s^2)

$$\frac{\partial w}{\partial t} + \vec{u} \cdot \nabla w + g = -\frac{1}{\rho}\frac{\partial p}{\partial z} + F_z$$

W/T	UW/L	g	P/Hρ	
~10^{-7}	~10^{-7}	~10	~10	(m/s^2)

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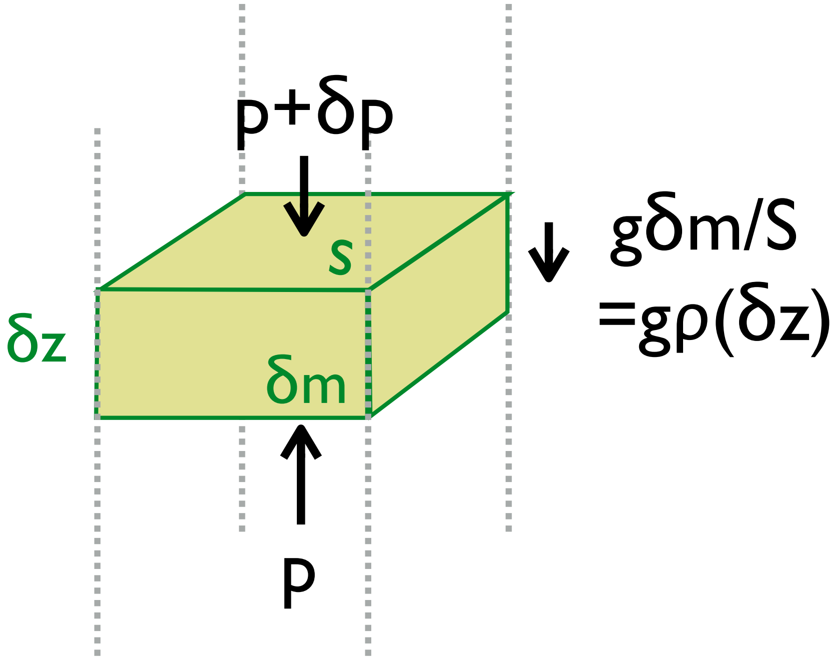
- P ~ 10^5 Pa
- H ~ 10^4 m

2. Hydrostatic balance (concept)

$$\frac{\partial p}{\partial z} = -\rho g$$

Physical concept

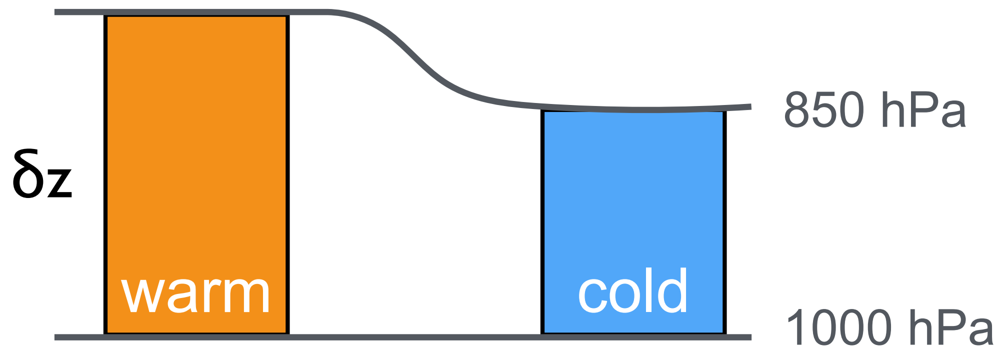


$p+\delta p+\rho g\delta z = p$

$\delta p/\delta z = -\rho g$

(Straightforward!)

Another application



$\delta z/\delta p = -RT/pg \quad (p=\rho RT)$

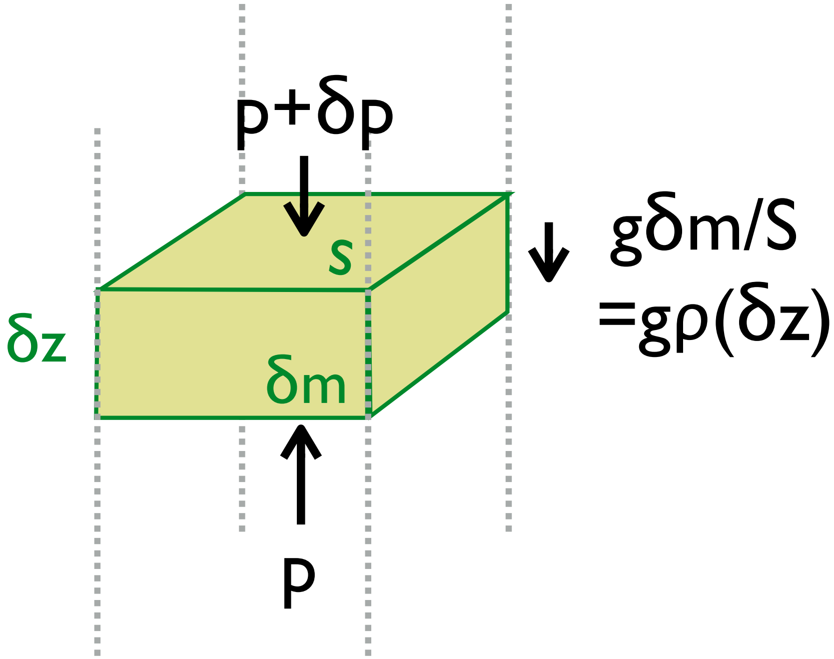
$\delta z = -RT\delta p/pg$

- Height of an air mass between two pressure levels is proportional to its temperature

2. Hydrostatic balance (concept)

$$\frac{\partial \Phi}{\partial p} = -\frac{RT}{p} \qquad \text{(P-coords)}$$

Physical concept

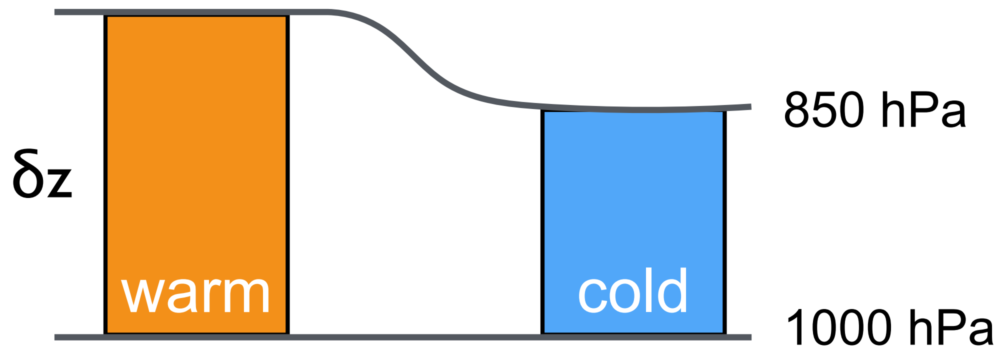


$p + \delta p + \rho g \delta z = p$

$\delta p / \delta z = -\rho g$

(Straightforward!)

Another application



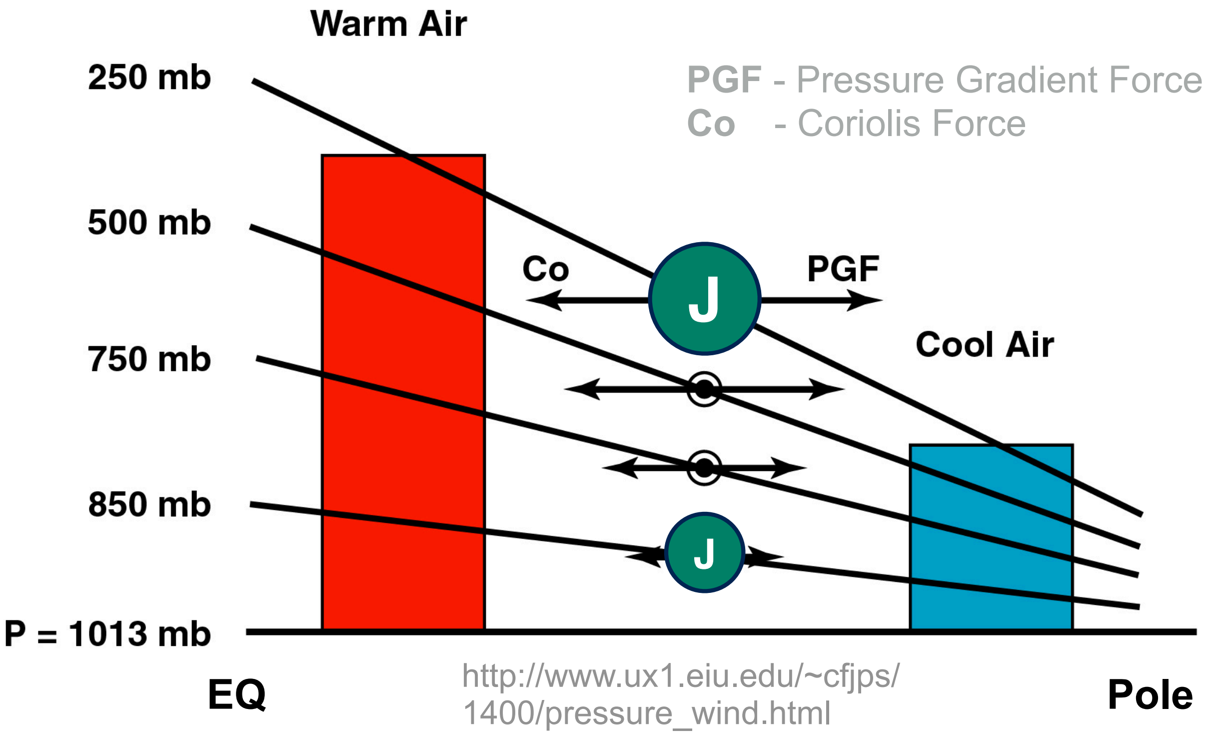
$\delta z / \delta p = -RT/pg \quad (p = \rho RT)$

$\delta z = -RT\delta p / pg$

- Height of an air mass between two pressure levels is proportional to its temperature

3. Thermal wind balance

Relationship between
vertical wind shear (**du/dz**) and
horizontal temperature gradient (**$\nabla_h T$**)



- Combination of **hydrostatic** and **geostrophic balances**

3. Thermal wind balance (derivation)

- Combination of hydrostatic and geostrophic balances

$$u_g = -\frac{1}{f\rho}\frac{\partial p}{\partial y}$$

$$\frac{\partial p}{\partial z} = -\rho g$$

3. Thermal wind balance (derivation)

- Combination of hydrostatic and geostrophic balances

$$u_g = -\frac{1}{f\rho}\frac{\partial p}{\partial y} \qquad \left(v_g = \frac{1}{f\rho}\frac{\partial p}{\partial x}\right)$$

$$\frac{\partial p}{\partial z} = -\rho g$$

- We use pressure coordinate for derivation (simple)

$$u_g = -\frac{1}{f}\frac{\partial \Phi}{\partial y}, \qquad \left[\text{using coordinate transform } \left(\partial p/\partial y\right)_z = \rho g\left(\partial z/\partial y\right)_p\right]$$

$$\frac{\partial \Phi}{\partial p} = -\frac{RT}{p}, \qquad (\text{where } \delta\Phi \equiv g\delta z)$$

3. Thermal wind balance (derivation)

- Combination of hydrostatic and geostrophic balances

$$u_g = -\frac{1}{f\rho}\frac{\partial p}{\partial y} \qquad \left(v_g = \frac{1}{f\rho}\frac{\partial p}{\partial x} \right)$$

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① ②

$$\frac{\partial}{\partial p}\left(u_g = -\frac{1}{f}\frac{\partial \Phi}{\partial y} \right) \Rightarrow \frac{\partial u_g}{\partial p} = \frac{1}{f}\frac{\partial}{\partial y}\left(\frac{RT}{p}\right)$$

3. Thermal wind balance (derivation)

- Combination of hydrostatic and geostrophic balances

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①

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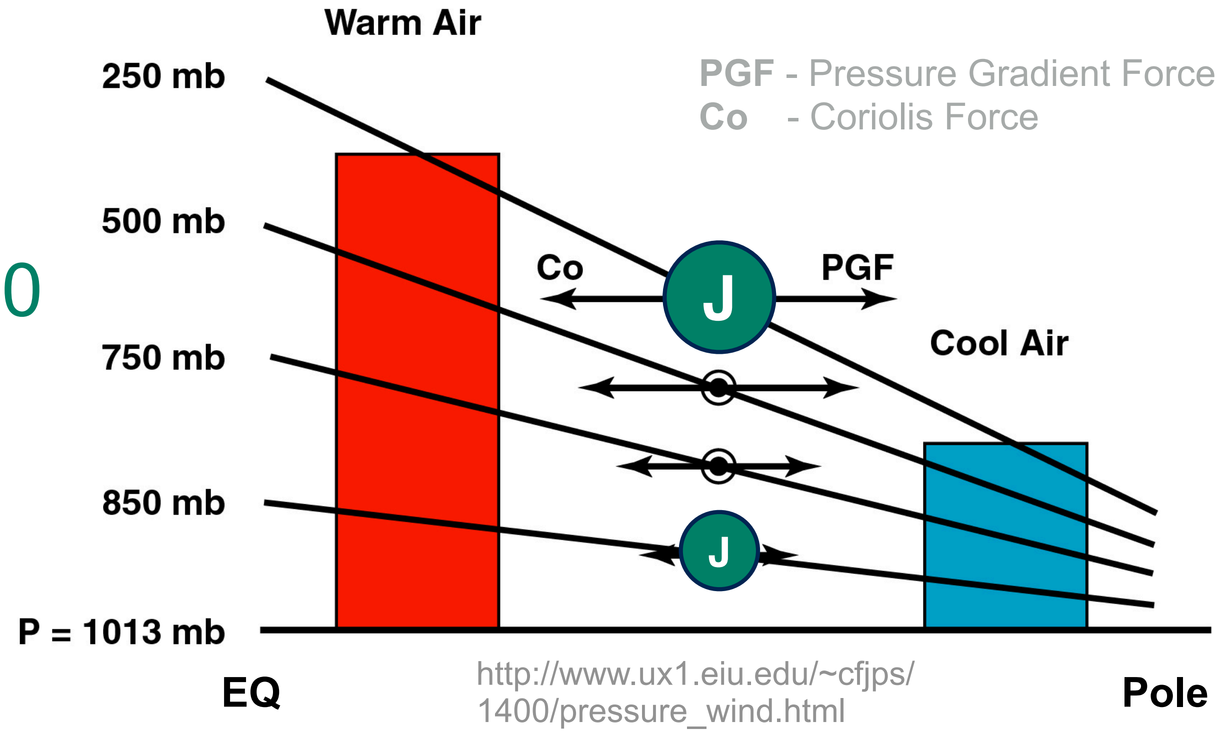
$$\frac{\partial u_g}{\partial \ln p} = \frac{R}{f}\frac{\partial T}{\partial y} \qquad \left(\frac{\partial u_g}{\partial z^*} = -\frac{R}{Hf}\frac{\partial T}{\partial y}, \text{ where } \delta z^* = -H\delta \ln p\right)$$

Thermal wind balance

log-pressure form (I love!)

3. Thermal wind balance (physical meaning)

$\partial u/\partial z^* > 0$
wind



http://www.ux1.eiu.edu/~cfjps/1400/pressure_wind.html

$-\partial T/\partial y > 0$
thermal

$$\frac{\partial u_g}{\partial \ln p} = \frac{R}{f}\frac{\partial T}{\partial y} \qquad \left(\frac{\partial u_g}{\partial z^*} = -\frac{R}{Hf}\frac{\partial T}{\partial y}, \quad \text{where } \delta z^* = -H\delta \ln p \right)$$

Thermal wind balance

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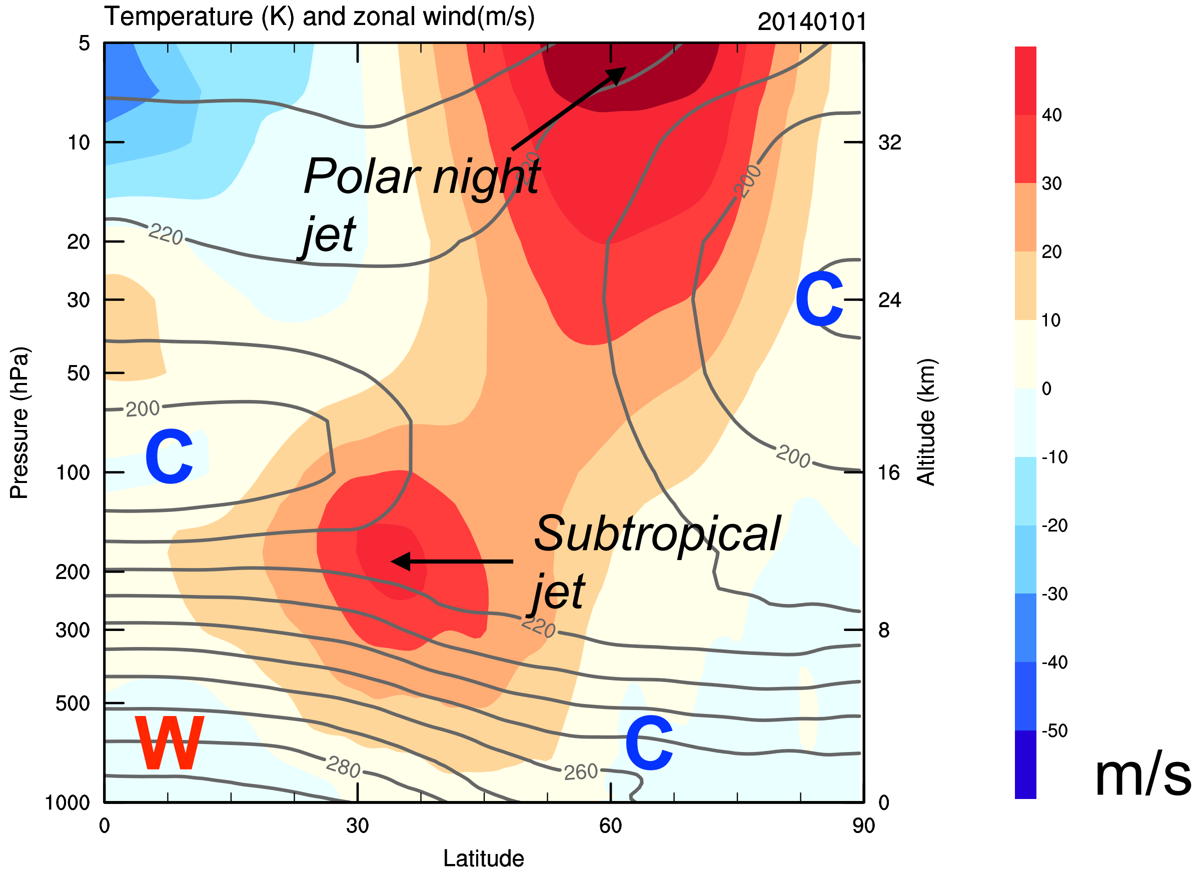
3. Thermal wind balance (real world)



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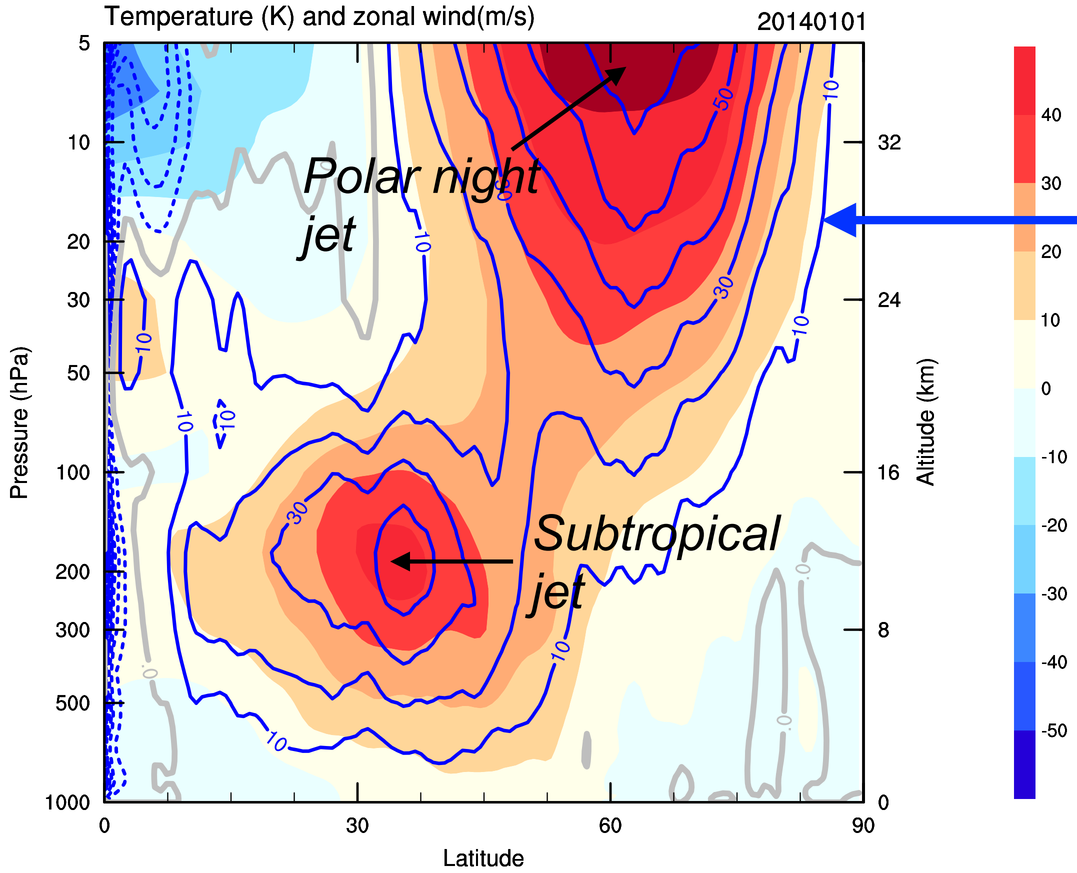

Ug from T

Not bad!

$$\frac{\partial u_g}{\partial \ln p} = \frac{R}{f}\frac{\partial T}{\partial y}$$

$$\left(\frac{\partial u_g}{\partial z^*} = -\frac{R}{Hf}\frac{\partial T}{\partial y}, \quad \text{where } \delta z^* = -H\delta \ln p \right)$$

Thermal wind balance

log-pressure form (I love!)

Recap

- Geostrophic balance
  Coriolis ≃ Horizontal pressure gradient
- Hydrostatic balance
  Gravity ≃ Vertical pressure gradient
- **Thermal wind balance**
  Geostrophic + Hydrostatic

$$\frac{\partial \overrightarrow{u_g}}{\partial \ln p} = -\frac{R}{f}\mathbf{k} \times \nabla_h T \qquad \frac{\partial \overrightarrow{u_g}}{\partial z^*} = \frac{R}{fH}\mathbf{k} \times \nabla_h T \quad (where, \quad \delta z^* = -H\delta \ln p)$$

Note:

1. Geostrophic and thermal wind balances work well at mid-to-high latitudes
2. Hydrostatic balance works at all latitudes